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ABSTRACT

Two phenomena related to the quality of student responses in regard to items used in a testing situation are involved in the construction of a pupil response model. The underlying phenomenon which defines the individual's cognitive limits is termed the hypothetical cognitive structure (HCS). The second phenomenon, which is a function of not only the HCS but also of experience in a content area, learning opportunities, and a person's past and present motivation, is termed the structure of the learned outcome (SOLO). The report looks at the optimum achievement of the HCS at the different developmental levels, then relates these to the SOLO levels. The reponse model is applied and summarized in a profile of response characteristics. Examples of pupils' responses in two particular elementary mathematical items are presented with commentary. Among the conclusions, it is stated that focusing the educator's attention on response levels and their structure has major implications for the classroom educational process. (MP)

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COGNITIVE DEVELOPMENT, MATHEMATICS LEARNING, INFORMATION PROCESSING AND A REFOCUSING

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
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Report from the Mathematics Work Group

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Preface

This paper is a revision of an essay prepared for a Seminar on Initial Learning of Addition and Subtraction held at the Wingspread Conference Center in Racine, Wisconsin, November 1979. A brief version containing some of the elements of this paper is Chapter 16, "The Structure of Learned Outcomes: A Refocusing for Mathematics Learning" in the book, Addition and Subtraction: A Developmental Perspective, T. P. Carpenter, J. M. Moser, and T. A. Romberg (Eds.), Lawrence Erlbaum Associates, Hillsdale, New Jersey, 1981.

Thomas A. Romberg
Editor

The Development of an Idea

In the first part of this paper, I will describe the evolution of my ideas about children's cognitive development and mathematics learning. I will trace the course of my thinking beginning with my days as a classroom teacher, when my task involved a series of practical problems to be solved, through a descriptive phase where I put the problems of mathematics learning into a Piagetian developmental framework, to an explanatory phase in which I related the developmental model to certain information processing concepts. Recently, in attempting to address the earlier research more directly to the practical problems faced by teachers and curriculum designers, I have turned to a response model and (with J. B. Biggs) have devised a taxonomy which is concerned with the structure of learning outcomes.

During the 1950s I taught for five years in elementary schools and five years in high schools in Queensland, Australia, and it was the experiences of those years which awakened my interest in making careful studies of the way children think. It was fascinating to see the kinds of mistakes that children made and to see the same mistakes repeated by different sets of children of about the same age, in different schools, and in different years. The similarity between the mistakes made in the same mathematical topic was not longitudinal, depending on who was teaching, but was rather a cross-sectional, across-teacher variable. This phenomenon was particularly noticeable in topics where children commonly had difficulty, for example, in the elementary school, in fractions, decimals, and long division. These

all seemed to produce, year after year, the same kinds of error. In the secondary school, the solution of simple, simultaneous, quadratic, and trigonometrical equations all produced characteristic mistakes year after year. What was even more remarkable for a young teacher was that no matter how one changed one's teaching method or textbook, the problem continued. Teachers in those days often tried to shift the responsibility for the mistakes to the children. The problem existed because the children concerned were inattentive, careless, or dull and thus, apart from tinkering with various ideas on classroom motivation, nothing much could be done.

This resolution of the problem was obviously unsatisfactory quite apart from the fact that some of the generalizations, such as carelessness, were patently untrue. Even with my limited experience at that time, it was clear that children were very rarely careless. Indeed, most seemed to take a great deal of trouble to try to follow the procedures they were taught. In solving problems children appeared to work very hard at thinking them through if they found them meaningful. Although their solutions (or teacher-required answers) would be incorrect, there was little justification for the label careless, in the sense of lacking care.

My interest in children's mistakes led, towards the end of the 1950s, to an interest in teaching children who, although not dull, were not succeeding at mathematics as taught in the classroom. This experience deepened my conviction that the way children were being taught mathematics was the reason for the kinds of mistakes which they were producing.

Talking with individual children about their mathematics made it clear that a lot of the common mistakes were due simply to the children having a misconception of the mathematical concept they were dealing with. This misconception was often easily derived from a standard classroom interchange. For example, the child in the early stages of learning the subtraction algorithm might be having trouble in getting the numbers the right way around and the teacher says, "Can't you see that you always take the smallest number from the biggest number?" The child adopts this rule of thumb. Some time later the child is still using the rule and getting into difficulties because he or she is unaware that the standard algorithm involves subtracting the bottom number from the top number. A further source of misconceptions lay in the way children were often taught certain topics. For example, after a very brief introduction to the concept of area and its measurement, the children spent a lot of time multiplying two numbers together to find the area of a variety of rectangles. When they regularly succeeded in this task, it was assumed that they understood the concept of area and its measurement. Only at a later stage when the children were given non-rectangular figures did it become apparent that their understanding of what they were actually measuring in terms of the product of the two numbers was not at all clear (Collis, 1969).

The kinds of experiences described in the last paragraph, along with the arrival in the early 1960s of the "New Math" program from the United States, stimulated my interest in investigating the problem much more closely. The New Math texts from groups such as School

Mathematics Study Group (SMSG), which presumably were to be used with normal children of 13 or 14 years, had expectations quite beyond what children would have been able to handle at that age level, in my experience.

At about the same time in the early 1960s, I was fortunate to meet up with G. L. Hubbard who had the same kind of interest in the learning difficulties in mathematics that apparently average children were having. Moreover, Hubbard had the resources available to explore this interest. A team was formed, consisting initially of Hubbard and Collis, and we arranged to teach experimental classes in certain private schools in Brisbane, Queensland. Most of the classes consisted of 30-40 girls. The classes ranged in level from the early primary school (most of the girls were about eight years old) to the middle secondary school level (girls about 14 years old). A team teaching technique was used in which one of the teams took the major responsibility for the classroom work for a particular lesson, or series of lessons, while the other observed. A person was also engaged as a recorder. This person's task was to record significant events in the classroom and to make general notes on the way in which the lesson proceeded.

In the initial stages of the work, we attempted to follow the children's reasoning wherever it led. Thus, we might introduce a topic with a particular plan of development in mind but find very quickly that the children's interest was in another, non-trivial, aspect of the topic. Instead of persisting with the original plan, we would switch to the line of thought the children were interested in and try to follow it through with them. For example, matchboxes

were introduced with a view to using them as informal measuring units. The children, however, became interested in the notation on the front of the matchbox which said "Average contents fifty." They wondered what this meant, so the succeeding lessons and experiences resulted in the children looking at the statistical aspects of "matchbox" mathematics. It was some few lessons later before the matchbox came back to be used for its original purpose of an informal measuring unit. The kind of teaching approach outlined above relied heavily upon our experience as teachers and our ability to see the structure of elementary mathematics as a whole.

It became clear, as the experimental teaching progressed with the various age groups, that primary school and early secondary school mathematics had to be structured so that the children could see the interrelationships between the various parts of mathematics for themselves if they were to get educational and practical value out of their experiences. In addition, it became very obvious that the children were able to reason with quite rigorous logic provided one did not exceed the level of logical functioning of which they were capable. This meant, in the case of the children in the experimental classes, that formal abstract logic had to be avoided and reliance placed on the concrete logic of classification, seriation, and equivalence as described by Peel (1960).

When the experimental teaching first started, we did not have any particular learning or developmental theory in mind. We were operating on the basis of our own experience in teaching children of these ages and our knowledge of the structure of elementary mathematics.

However, it became clear after several months of experience with the children that this base was not sufficient, especially as the mathematical structure and the technique used to develop it seemed to interact substantially with the children's cognitive functioning. In order to know where the project was headed and what needed to be done next, it was necessary to find some kind of theoretical framework upon which to base decisions on both teaching technique and program components.

After considerable reading, thought and debate on the matter, we decided that the most satisfactory theory, as far as teaching was concerned, was one based on the Piagetian model because it seemed to take the most account of the children's form of logical reasoning. For the structuring of the program, Ausubel's theory of the acquisition of meaningful verbal learning (Ausubel, 1963) was linked with the Piagetian model (Collis, 1970). This composite model was used to plan the further teaching of mathematics to the experimental classes.

From the work of those years, there were two main outcomes. First, a series of texts was written by Hubbard between 1963 and 1965. I contributed a number of teaching notes for the teachers' editions of these texts. Unfortunately, the texts (Hubbard, 1963), 1964, 1965), which were designed for the early secondary school, are now out of print but they were used in a number of Queensland schools for some years after their production. The second significant outcome was a formulation of a general theory of teaching mathematics to children from about eight to fourteen years of age. This formulation has been described in various places (e.g., Collis,

1969; Hubbard, 1971). In addition, the broad summary of the theory is set out in Collis, 1975a.

By the mid 1960s, Hubbard and I were beginning to find divergent interests. Hubbard wanted to continue with the same experimental teaching technique and use the insights obtained to design curriculum materials and publish texts. I had become more involved in the ways in which children were thinking, particularly with mathematics items.

In 1968 our team split up. I accepted a position at the University of Newcastle (New South Wales) to do research in cognition with the psychology department, using mathematical items and the Piagetian model as a theoretical framework. This turned out to be a happy combination. The mathematics educators saw the items we had developed in the course of solving the problems of teaching mathematics as having direct relevance to the classroom and the programming of mathematics courses; the use of these kinds of items was new to most of the psychologists I came in contact with and thus threw a different light on their views of problems of cognition.

In the late 1960s and the early 1970s, I conducted a series of experiments with children between the ages of seven and seventeen years with the object of teasing out the Piagetian developmental stage levels, or the constructs underlying these, using items involving elementary mathematics. The tests were both group and individual. The former were intended to provide normative data on a cross section of children from seven to seventeen years. The latter were more in-depth clinical studies meant to track down the way the children were arriving at their results and what this implied about their

level of logical functioning. The results of these studies are published in a number of places (Collis, 1969, 1970, 1971, 1972, 1973, 1974, 1975a).

It seems appropriate to summarize the results of this part of the work, however briefly, at this point. The series of studies clarified some of the basic Piagetian concepts and defined the different Piagetian stages operationally in terms of mathematics items. The differences between operating at different stage and sub-stage levels, it was found, could be described in terms of the following constructs which obviously are not disjoint but interact with one another:

1. Complexity of mental operation involved: This refers to what the child has to retain in the working memory while he or she deals with the problem to be solved. For example, to solve $4 + 3 = \square + 4$ does not require as much "mental effort" as $7 - 4 = \square - 7$. The former is most frequently done correctly by the younger children by using a pattern completion strategy which makes minimal cognitive demands; the latter requires at least two closures and a decision to add where subtraction is strongly suggested by the form of the item.

2. Abstractness of elements involved: The increasingly abstract nature of numbers as they become larger is a good example of this notion. Small numbers, less than 10 for example, are very real to the child in the early elementary school, but large numbers such as 289 are quite abstract and hence without substantive meaning. The young child can not visualize what such a large number represents. Even further removed from reality in the abstraction process is the use of letters to represent

variables; the ability to work meaningfully with this concept does not appear until well into adolescence.

3. Ability to handle abstract systems: This refers to the ability of the child to solve problems given a set of rules and definitions; the defined elements of the problem together with the set of rules have no necessary reference to the child's physical world. Many topics in arithmetic where children are asked to deal with elements that are not within their experience are of this kind. The topic becomes a set of arbitrary rules which are applied to abstract elements. For example, topics like stocks and shares, bank interest, profit and loss, commission and so on are almost custom-built for this kind of criticism although our methods of teaching can arrange for any mathematical topic to conform to this model.

4. Ability to operate on operations: The use of the inverse in solving equations can be used to illustrate this notion. Prior to the formal level the children do not use the inverse in a reciprocal manner, controlling an operation and maintaining a balance in the system at the same time. Instead they tend to use a negating mechanism which is related only to that part of the system upon which they are presently focusing. For example, in solving the following equation, $x + 4 = 9$, the child using the negating mechanism will reason, "minussing 4" undoes "plussing 4" therefore, if $x + 4 = 9$

then $x = 5$

and will not see any point in using any intermediate steps.

The child who uses a reciprocal strategy will, if pressed, see the value of several intermediate steps, the first one "controlling" the operation while keeping the original statement in view and in balance, thus: $x + 4 = 9$

$$x + 4 - 4 = 5 - 4$$

$$x = 5$$

5. Acceptance of lack of closure: This construct refers to the level of the child's ability to work with operations without the necessity for closing the operation. In early primary school, this ability is not very well developed at all. The child insists on closing an operation such as 2 + 3 immediately before any further consideration is taken; at a slightly later stage the child is able to deal with the operations and reason with them as long as there is a guarantee that the operation could be closed to a unique result at any particular time. At the highest level of functioning the adolescent is able to resist closing the operations and keeps them entirely open as long as necessary to come to a logical conclusion.

6. Multiple interacting systems: This particular construct was explicated by Lunzer (1973) when he was distinguishing between simple and complex systems. In the present context it refers to the child's ability to handle mathematical formulas at different levels of sophistication. For example, at the junior high school level, most children can use a formula such as A = L x W by realizing that, given the dimensions, this formula will enable a measure to be made of the area of any

rectangle. This represents essentially a simple system of covariation; the area changes as the rectangle changes and "L x W" changes as the rectangle changes. What cannot be done at this stage is to relate changes in one or more of the variables A, L, and W to changes in one or more of the others. (For example, "If A is to stay constant and W is to be changed in some way--doubling, taking a fraction of the original--what must be done to the length L to satisfy these new conditions?")

The research up to this point (the early 1970s) gave a useful description of the stages of cognitive development of the child using items from elementary mathematics. However, no investigator could be satisfied with a mere description of how things worked, no matter how useful the descriptions were in terms of making curriculum and teaching decisions. My next step (about 1974) was to begin looking for an explanation of the stage phenomenon.

An Explanation of the Stage Phenomenon

During some early studies an interesting phenomenon appeared. Early elementary school children seemed capable of working meaningfully with mathematical items which involved two elements and one operation (for example, 3 + 4) but they seemed unable to work successfully when a further element and operation was introduced (for example, 2 + 3 + 4). Success with this type of item came a little later. When tested individually the younger children appeared unable to retain all the necessary information long enough to process it.

A typical interview with a seven year old went like this:

Experimenter: What number does 2 + 3 + 4 equal?

Child: 2 + 3 = 5 and (pause) What was the other number?

Experimenter: I said what number does 2 + 3 + 4 equal?

Child: Oh yes! Now, 2 + (pause) what is the sum again?

A similar pattern of responses was obtained with more complex exercises through the higher stages up as far as formal operations. For example, in another study an eleven year old girl was given the information that $y = 365$

and that, $y + 289 = 289 + 365$,

and is asked to say whether the last statement is true or false and to give a reason. The child showed every sign of being confused by having too much information to take into account. In the case in question she decided to add up the two numbers on the right hand side, 289 and 365. Having worked this out she then alternated her attention between this calculation and the left hand side of the equation muttering over and over again the statement, "y + 289 equals." She ignored altogether the information, y = 365. She finally became completely confused and gave up. The same kinds of symptoms can be demonstrated with older children of fourteen or fifteen years if one makes the questions more complex still. The children in each case show signs of cognitive strain and repeat parts of the problem aloud. They seem to indicate a difficulty in bringing together the relevant bits of information long enough to allow them to be processed. In all

cases, protocols for the individual students showing this problem indicate that they are behaving as if they are exceeding their particular capacity to process data.

Let us examine a possible model to explain this behavior. Suppose the circle below represents the actual space available for processing data. Let us look at the child's problem at the early elementary school stage and see what may be happening. First of all the child is asked

to enter into the processing space the number 2, 2 being to the child not an abstraction, a number that exists in itself, but two real things that have to be kept in the mind's eye, as it were, to give meaning to the symbol "2." Likewise the next piece of information is an operation "plus" which doesn't have an existence of its own but also has to represent some physical act of putting together. Then comes the "3" which also consists of three things and must be treated like the "2" already entered. These three pieces of data come together and give us the total 5 which the child attempts to retain in the working space together with the "2" and the "3" and the "plus." The diagram suggests that the space is now fully occupied. Any attempt to add further data results in what is called an overload in calculator terms. When this overload occurs putting in further information means that some information has to go out.

A year or so later at the next substage of development the child is able to cope with this kind of problem. Why? Evidence from individual interviews suggests that the difference lies in the fact that, by then,

the small numbers, like the "2" and the "3," and the operation of addition have become entities in their own right. The child does not have to keep them in the mind's eye or relate them to some physical phenomenon - they can be treated as things in themselves and thus occupy very little working space. This means the child at this substage can handle several small numbers and a number of operations by closing in sequence. This kind of explanation fits all the aspects of the situation I have observed: the oral repetition of the data, the continual refocusing on different parts of the data, and the physical symptoms of strain and confusion.

There are several models, differing in detail but compatible overall, which not only support these intuitions but also satisfactorily explain the stage development phenomenon. Two of these models were presented by their authors Halford (1977) and Pascual-Leone (1977) who participated with me in a symposium on stages in thinking held in Pavia (Italy) in 1977. A third model, the one that I wish to use in this paper, is one presented by Dr. Robbie Case of the Ontario Institute for Educational Studies. However, we will come back to the Case theory a little later in the paper.

A Change of Focus

A Response Model

So far this paper has outlined the genesis of my ideas from the time of typical classroom experiences through the development of constructs which described the phenomenon in terms of mathematical items

on to a possible explanation for the phenomenon in terms of the information processing model. I will now describe my more recent thinking in the area.

About 1975 I set out with Professor J.B. Biggs of the University of Newcastle (N.S.W.) to gather together the substantial amount of work which had been done in the classroom on the Piagetian notion of stages of development. In particular E.A. Peel and his students at the University of Birmingham had done a lot of work in content areas as diverse as geography, history and English literature. Other writers such as Shayer of the University of London had done work in science. Biggs and I set out basically to find examples of levels of cognitive development in the major teaching subject areas at high school and late primary school so that these examples could be made available to teachers. We hoped, for example, to help teachers recognize student errors as developmental phenomena rather than merely as carelessness.

We also planned to produce some sort of a checklist for teachers so that they could code their own students' responses in terms of developmental level. However, as soon as the literature review had begun and, more significantly, when we began giving some tests of our own to build on the available data, several problems surfaced, three of which are outlined below.

1. The criteria used by the various researchers for determining different developmental stages varied considerably across content areas. Inverse operations and multiple interacting systems, for instance, while quite clearly useful in logico-mathematical tasks, did not seem as significant

in other areas of content, such as in English or history. This does not mean to say that they could not be recognized there but rather that the authors who were responsible for the studies in these areas did not seem to take them into account. On the other hand certain areas in English such as "quality of expression" seemed to be organized in terms of criteria which would not readily apply in science or mathematics.

2. When testing subjects on items from different content areas we found that the well known Piagetian concept of décalage was very much the rule rather than the exception. This finding seemed at odds with both the developmental approach and the idea of developmental stages. The same student on different occasions would vary three and four stages in the same content area. Age ranges for typical responses seemed too gross a measure to accommodate the traditional stage development scheme.

3. Another puzzling feature which arose was inconsistencies in the same student's response upon retesting on the same item. A student might respond at what would be termed a middle concrete level and then, when retested some time later, give responses one or two levels higher (or even lower) than the level at which the material had been previously encoded.

Considerations like these called into question the idea of categorizing students into developmental levels on the basis of their responses to particular items in particular content areas. Thus, the focus had to shift away from a response implying a stage of development to consideration of the quality of each individual response per se. It appeared

that two phenomena were involved, not one as had been supposed. The first, underlying phenomenon which defines the individual's cognitive limits might be termed the hypothetical cognitive structure (cf. Piaget's Stages of Cognitive Development). The second, which would be a function not only of the hypothetical cognitive structure but also of experience in a content area, learning opportunities, and the person's present and past motivation might be termed the structure of the learned outcome (SOLO) (Collis & Biggs, 1979). The former might be likened to the old idea of the IQ which was a measure of intelligence and considered permanent. The latter may be considered to be like achievement on a particular test at a particular time. Although the two constructs are closely related, the former is relatively fixed and is virtually immeasurable and presumably unalterable (at least in the short term) by what teachers do, while the latter is flexible and measurable by the teacher. In fact, it can be used by the teacher to guide the design of both lessons and programs.

Let us look briefly at the optimum achievement of the hypothetical cognitive structure (HCS) at the different stages of development and then relate these to the SOLO levels.

Pre-operational HCS Stage: This is the stage that in the Piagetian model is nonlogical or prelogical. These children typically cannot conserve and so mathematics as such is beyond them but there appears to be real value in encouraging them to work at premathematical exercises. The equivalent level in the SOLO Taxonomy is called prestructural. Here the responses typically indicate that the child has no real feeling for what

one would call mathematics; very often the responses they give to mathematical tasks are irrelevant or tautological.

The early concrete operational HCS: An elementary basis for a concrete logic of classes, differences and equivalences now exists. Conservation, which involves elementary use of the reversibility principle, seems to be well established and there is a basis upon which to develop some sort of logical, even mathematical, structure. The equivalent level in the SOLO Taxonomy is termed unistructural. At this level children asked to form a conclusion on the basis of given information will select one piece from all the given data and immediately come to a rapid conclusion. In mathematics they demonstrate a necessity to close any operation quickly and they find it difficult to find meaning in expressions that have more than one operation with small numbers. In solving any problem they tend to go forward from the starting point and then only one step. For example, in solving the equation $y + 4 = 7$ they will give the response 3; the reason involves some sort of counting on procedure. Typically they will not see that subtraction is a useful procedure for solving the problem.

Middle concrete operational HCS: This is a period of well established concrete logic of differences, classes and equivalences. Children at this stage do not see interrelationships or, at least, do not consider interrelationships in the data. Reversibility of operations is available to them. However, they demonstrate that their reasoning is clearly confined to the physical world of the "here and now," despite

indications of a raised level of abstraction in that they can use more operations, larger numbers, and more propositions in verbal problems, the equivalent SOLO level, the multi-structural response, indicates that the child comes to a conclusion on the basis of a sequence of discrete pieces of information selected from the data. In mathematics, sequences of closures are used meaningfully if the numbers are small, although if large numbers are involved the number of meaningful relations or operations that can be handled is consequently reduced. The child seems to lack an overall view of the interrelationships between the operations and elements in a statement. It is as if the statement represents a series of instructions to be performed in sequence.

Concrete generalization HCS: This is the high point of concrete operational logic. The most significant feature is perhaps an ability to generalize from several concrete instances, although no abstract hypotheses are considered and thus the generalizations are often inadequate. The student indicates an ability to interrelate given data but not to go outside it. The equivalent SOLO level is called relational. At this level the child's response indicates an ability to relate one part of a system to another in a quite concrete way. For example, in terms of numbers the child is able to deduce that $\frac{273 \times 471}{471}$ is equivalent to $\frac{384 \times 273}{384}$ but does not realize that this is an example of the abstraction $\frac{a \times b}{b} = \frac{a \times m}{m}$.

The line of reasoning can be easily verified by asking the child to prove that one statement is equivalent to the other. Typically, the proof will involve working out the numerator and dividing by the denominator in order to achieve a unique result which happens to be the same for both statements. Responses at this level indicate that the child can keep track of the key interrelationships within a given numerical statement.

Formal operational stage HSC: This is clearly associated with mature formal logic. The adolescent at this level of functioning has the ability to take all the data and their interrelationships into account after obtaining an overview of the problem and considering an appropriate abstract hypothesis or generalization. This hypothesis or generalization is tested against the given data and against other considerations which may not be given but which are germane to the problem. The equivalent SOLO level involved is the extended abstract response. This level of response reveals an ability to deal with complete abstractions so long as they are within a well defined system. This last is the hallmark of this level of response. Variables are no problem and do not need reference to physical analogues; mathematical and logical operations take on a reality of their own; balanced systems such as equations can be overviewed and manipulated so that the system remains in balance.

Information Processing and the Response Model

Let us now consider how this response model fits the information processing paradigm which was earlier hypothesized to explain the stage

development phenomena. We will use the Information Processing Model put forward by Dr. Robbie Case (1978).

Case's theory fits the general Piagetian model with regard to both the content and process of development. His major change concerns the way the general developmental factor is conceptualized. It is conceived as a quantifiable level of working memory rather than as a general notion of "operativity"; even this change is not incompatible with the Piagetian model. Case's basic position can be summed up in the following five postulates:

1. that children pass through a series of sub-stages within each major stage (of development) in which their strategies or rules for approaching the problems characteristic of that stage become increasingly complex;
2. that one necessary condition for strategy evolution is exposure to information of relevance to the specific domain in question;
3. that a second necessary condition is the acquisition of sufficient working memory to coordinate the information of relevance to the more advanced strategy;
4. that working memory capacity is constant but that there is a gradual increase in functional working memory within each stage, due to the automatization of the operations which are characteristic of that stage;
5. that once this functional working memory reaches a critical level within each stage, the way is paved for the assembly of the

higher order operation which underlies the strategies for the next stage. (Case, 1978)

These postulates clearly refer to an individual's general level of cognitive development, not to the structure of a specific response in a particular content area at a certain point in time. Equally clearly the two domains are closely related. Especially in relation to the learning of elementary mathematics, it would appear useful at this stage to link the Case postulates to level of response. In teaching mathematics or in organizing the mathematics curriculum one is dealing with the structure of the learned outcome rather than with the individual's general cognitive functioning. One has to assume that the basic substrate strategies about which Case is concerned are developing and use them to determine the upper level of the response expectations. One must be aware, of course, that what is being done in mathematics is probably affecting the basic level of general cognitive functioning.

Let us examine the postulates in relation to the response model outlined earlier, keeping in mind that one of the important functions of the teacher of elementary mathematics is to encourage students to respond at the highest possible SOLO level.

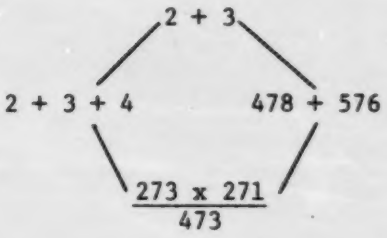
Postulate 1 can be seen operating throughout the increasing complexity represented by the continuum from Unistructural to Relational responses in the area of numbers and operations. For example,

(a) each level of response relies upon closure at some concrete level but there is a steady increase in the complexity of the way in which the concept is used;

(b) the only mathematical operations which can be handled successfully are those which can be based on some concrete analogue such as the four operations of elementary arithmetic;

(c) there is a gradual increase both in the abstractness of the elements able to be used with understanding and in the number and interrelatedness of the operations which can be meaningfully used with those elements.

The following diagram may make our point more clearly:



Unistructural - one operation, closed immediately

Multistructural - multiple operations closed in sequence

Relational - multiple operations and their interrelationships seen within an expression

The different levels of response are cumulative. Each one adds something to the previous one, until the greatest complexity attainable at the particular stage of development is achieved. The developmental stage spanned by the example given is the period of concrete operations and covers the SOLO levels from unistructural through multistructural to relational.

Postulate 2 bears on the fact that to expect higher level responses in mathematics must require that the individual has extensive experience with both the content and process of mathematics. The diagram above illustrates this quite well. A child needs extensive experience in responding at the unistructural level in order to ensure the numbers and the basic operations become real (i.e. concrete) things in themselves.

In other words, extensive practice is needed to achieve a degree of automatization which will allow movement to the next level of responding. At this next level a number of operations can be closed in sequence and, within limits, large numbers, those beyond the child's immediate ability to visualize the object involved, can be handled with understanding.

At this stage academic subjects are usually taught with two main effects on the student in mind: the assimilation and understanding of the content of the subject (the facts and concepts that constitute knowledge of the subject), and the cognitive processes that are induced by a proper understanding and application of the subject (the skills and strategies that constitute the appropriate way of thinking for that subject). Bruner (1960) strongly emphasized the interplay of content and process features in the overall structure of a subject matter. He spoke of the "generic codes" of a subject. These are the basic processes and content structures that make the subject. In the present context this means that, right from the beginning of their contact with mathematics in school, children need to experience the structure and rigor of mathematics presented at a level which matches their intellectual capacity by a teacher who has a feeling for both the "generic code" of mathematics and the level of functioning indicated by a child's responses.

Postulates 3 and 4 relate to the availability of working memory space to permit more advanced strategies within a particular stage. The relevance of this notion in the present discussion can again best be seen by referring to our diagram. If the working memory capacity is constant, keeping track of the numbers and the operation involved occupies almost

all the available space for a person confined to the unistructural response level. The "2" and the "3" are not yet independent entities. As described earlier the "2" for example must be visualized as two physically available objects. Likewise the addition operation must be related to some physical union. A child working at this level of response indicates cognitive overload by losing track of part of the question when presented with an item involving multiple operations. As also mentioned earlier, similar patterns of response are obtained with more complex exercises at higher levels of functioning.

Postulates 3 and 4 indicate that to produce a higher level response within the logical domain characteristic of the major stage level, there must be an increase in functional working memory which comes about through automatization of the elements and operations involved. In the present context this means that many experiences with small numbers in conjunction with one operation (e.g., 2 + 3 = 5) enable the individual to begin to regard the numbers and operations as entities in themselves without immediate reference to some physical component. The need to use some of the working memory space to monitor this last component having been dispensed with, functionally there is more space available for additional elements and operations and thus multistructural responses become possible (e.g., 2 + 3 + 4 = 9). Similar progress makes possible the development of relational responses. Referring again to the diagram we can see that the move to relational level responses depends upon having automatized the concept of numbers and the operations on them so that one

can regard big numbers as being as real as small numbers and see that closure of the operations is available if necessary. Typically relational responses reveal an ability to stand off and have an overview of how the operations within any given number statement might be interrelated.

Postulate 5 is concerned with the move between major stages or, in terms of the response model described here, between responses at prestructural and unistructural levels and again between relational and extended abstract responses. If we take the latter transition as an example, all the literature points to a distinct shift in the quality of response. The quality of the structure of the extended abstract responses makes it reasonable to suggest that the functional working memory is operating at maximum capacity and that the underlying logical functioning has taken a giant step forward. Examination of extended abstract responses shows that improvements which have previously taken place within a stage because of automatization are now extended and articulated into a comprehensive and highly efficient system which allows for the application of much higher level strategies. For example, the ability indicated in relational responses to handle multiple familiar operations with large numbers and to see relationships between the operations within an expression where closure is available at any time is extended and generalized to an ability to handle defined operations with variables and to see relationships between the operations within an equivalence statement.

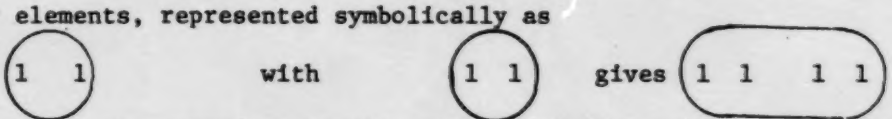
The Response Model in Use

Elementary Mathematics

The difficulty in making a direct application of the response model

in mathematics lies in the fact that the actual response made by the child does not usually indicate the complexity of the thought process which gave rise to it. For example, if a child is asked to find the value of "y" in "y + 5 = 9," and responds correctly with the answer "4," one does not know whether this has been achieved by the simple unistructural device of counting on or by an extended abstract approach which would involve general principles such as inverse operations, associativity, and so on. To find the complexity of the structure used by the child further probing is necessary. The summaries of the response characteristics which follow assume this further probing.

1. Prestructural Responses. A response at the prestructural level shows that the respondent is not deploying the thought structures to handle numbers and their "mathematical" combinations meaningfully. That is not to say that a child responding at the prestructural level could not answer the question: "What does 2 and 2 give?" However, being able to respond "4" does not necessarily mean that the individual possesses the knowledge or the experiences to be sure that a group of two elements combined with a further group of two elements always gives a group of four elements, represented symbolically as

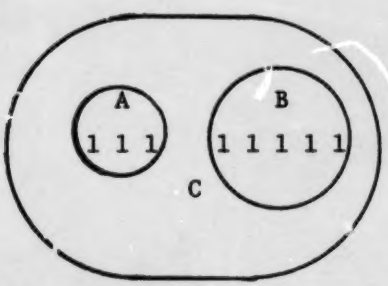


Being able to respond correctly to the task of matching a set of numerals with a set of four objects to be counted need not imply that the child understands the number idea four, which incorporates the notions of invariance and conservation. For instance the child might well be able

to use the counting sequence accurately up to a certain point but still give nonconserving responses to some of the classical number conservation tasks.

Prestructural responses often indicate that the individuals concerned are developing bases for operational thinking, but it is not until the next response level that one sees the thought structures necessary to handle number concepts, groupings, pairing corresponding elements, and so on.

2. Unistructural Responses. Responses at this level show that the respondent can only work with elements based in immediately observable physical experience (e.g., 7 but not 759). The operations of ordinary arithmetic as well are related directly to verifiable experience.

e.g. 

→ n(C) = n(A) + n(B) → 8 = 3 + 5

Responses at this level to items using either large numbers or several operations in sequence tend to indicate that the child does not find the question meaningful. The respondent needs to see that a unique result exists, "real" in terms of the student's reality. This need may be termed a requirement for closure, and is related to a similar need in other content areas to come to a quick firm decision on the basis of

one datum. The idea of "inverse," if incorporated in responses at this level, is physical. For example, subtraction is seen as "what is put down can be taken up." No concept of the inverse as a process exists; the typical unistructural response to the question of why $x = 3$ in the statement $x + 4 = 7$ is to use a counting on procedure of some kind. If pressed further most respondents confined to this level will deny that "$x = 7 - 4$" is a legitimate method of solving the problem.

In summary then unistructural responses are marked by a single direct relationship to criteria that are concretely available (either physically or iconically).

3. Multistructural Responses. Responses in this category are marked by a student's apparent ability to handle logical operations provided they can be applied directly to particular experiences. Operations are not related to one another nor to abstract systems set up independently of physical experience. Thus, in working with addition and subtraction operations, the pupil responding at this level regards the result of each operation as unique, and can cope well with a statement which involves a sequence of discrete closures as long as there is no necessity to keep track of relationships among the operations in the statement. The arithmetical operations seem to be seen as reflections of reality - the sums and differences refer to actual sets of objects whether present or not.

At this level of response the pupil still tends to work with qualitative comparisons like "the closer" or "the larger." Thinking depends

on, and is bound by, the individual's perception of what is real and a marked inability to set up an empirical system based on measurement. Children responding at this level are beginning to perceive consistency between their qualitative comparisons which at times leads them to consider quantitative comparisons, but using additive procedures only. Thus their responses show that ordering elements by using direct excess comparisons is a common strategy, but ordering or comparison by ratio is not.

Meaningful use of the four operations of arithmetic shows up only if the uniqueness of the result is guaranteed by previous experience with both the elements and operations. A number of elements may be considered if they related to direct physical experience. However, if the elements are more abstract, only a very limited number can be handled. This means that a pupil responding at this level will cope with items like 3 + 7 + 4 because the elements are verifiable and the 3 + 7 can be closed so that it gives still another verifiable element, thus allowing closure with 10 + 4 = 14. A second type of item that can be handled is 475 + 234; here the operation is familiar and thus the process can be carried out because the pupil knows from previous experience that a unique result will always be found.

Closure is still the ultimate guarantee of uniqueness and the ability to handle multiple operations with small numbers by a series of meaningful closures may be seen as analogous to using a sequence of given propositions to support a particular judgment in other content areas.

The concept of inverse which comes through at this level of response is that it is a "destroying" process. Although, in the classroom, this kind of thinking is difficult to distinguish from thinking where the inverse is seen as an "undoing" process, the student's concept of destroying possesses an irreversible quality. For example, the student regards subtracting as destroying the effect of addition without specifically relating to the operations themselves. To find the value of "y" in "y + 4 = 7," "y" is regarded as a unique number to which "4" has been added; subtracting "4" happens to destroy the effect of the original addition. Probing usually reveals that the child explains the correct response to the question, not by referring to the addition and subtraction operations, but by adopting a more primitive strategy, counting or saying the equivalent of, "that's how I was taught."

4. Relational Responses. Responses at this level seem to rely upon the same basic skills at the multistructural response level; the major advance appears to be less reliance on seeing uniqueness in the results of operations even though there is still a requirement for uniqueness of outcome to be guaranteed. This guarantee of closure is obtained by making a generalization from concrete instances. The individual response is still bound to empirical evidence but the respondent is now prepared to infer beyond what can be demonstrated by model, and to form a generalization from a number of specific cases. Thus at this level, where appropriate, the response takes the form of a concrete generalization where a few specific positive instances guarantee the reliability of a rule. Typically the responses indicate that the student is looking for

positive instances to form a generalization, but is unaware of the need to check for counter-examples and constraining conditions. Moreover, as the response is based on the immediately available empirical evidence only, the student is unable to consider hypothetical instances.

The basic ability to handle a statement involving a series of closures is now generalized in two ways. First, the size of the numbers used rarely causes problems; second, and more significantly, the child now indicates an ability to keep track of the relationships within the given statement.

Relational responses show an ability to handle generalized (apparently abstract) elements based on a few specific verifiable instances, e.g., 2a + 3a = 5a. Probing reveals, however, that the uniqueness of the result (the requirement of closure at this level) must still be guaranteed. In the example just given, questioning reveals that the truth of the statement most likely rests on some concrete analogue such as "2 apples together with 3 apples gives 5 apples" rather than on any satisfactory abstract mathematical generalization.

In responding to questions related to statements such as the formula y = m x s x t these students' responses show that they rely on the following facts: (a) that each letter stands for a unique number on any particular occasion, and (b) that each binary operation involved can be closed at any stage.

At this level the following types of items will elicit correct responses:

(a) if 9 * 4 = 5 and 7 * 2 = 5,

what operation does * stand for?

(b) if 6 * 2 = 12 and 5 * 3 = 15,

complete 4 * 5 =

(c) decide whether the following pairs of expressions are equivalent:

$\frac{259 \times 416}{259}$ and $\frac{376 \times 416}{376}$

(469 + 361) - 257 and (468 + 362) - 254

At this level of response the "inverse" process becomes an "undoing" of an operation previously performed. Thus it possesses a reversible quality, but is limited by the fact that it can only be applied to familiar operations. For example, the student may solve simple equations by using the property that addition undoes subtraction, multiplication undoes division and vice versa, but is not necessarily able to see that squaring undoes the square root in a simple surdic equation.

Individuals responding at this level appear able to use an abstract rule that they have obtained from generalizing a number of specific instances but they do not demonstrate the cognitive structures necessary to use the rule in a slightly different situation. To take an example used in an earlier section of this chapter, a student may derive the formula for the area of a rectangle (A = L x W) by generalizing from a number of particular examples, and then appear to use the formula as if it now formed part of an abstract system. However, if a question is given that requires seeing A, L, and W as variables instead of allowing

A, L, and W to be seen as unique, each representing a particular number on a particular occasion, then the typical response at this level shows an inability to handle the problem. An example of this latter type of question would be: "If a rectangle has length twice its breadth and its area is 72 square units, find its dimensions." The difficulty for the student confined to the relational level is that in a formula such as $A = L \times W$ each element must represent a specific, unique number and a closed result must be guaranteed as soon as a substitution is made. This necessity for a guarantee of immediate closure which typifies responses at this level inhibits the ability to overview the data and comprehend the abstract relationships between variables.

In conclusion then responses at the relational level are given on the basis of concrete generalizations where a few specific instances satisfy the respondent of the reliability of a rule and where the result of an operation, even though it be on apparent variables, is necessarily considered to be unique. In other words, the individual relates elements within the immediately available concrete system and forms generalizations on this basis.

5. Extended Abstract Responses. At this level responses show that students reason and consider ideas at the abstract level without having to rely on experiences in the physical world. Their answers are clearly based on deductive reasoning from a carefully chosen hypothesis or premise. Abstract variables which may have a bearing on the solution to the problem are visualized and manipulated; responses are no longer bound by concrete experiences. Respondents are no longer satisfied that one or two positive

instances are a sufficient basis from which to generalize; a comprehensive inductive procedure is invoked. The student responding at this level does not require operations to be actually closed; closure and uniqueness are looked upon as abstract properties that have certain implications if they are available. The ability to operate on operations is in evidence (at the earlier levels the responses relate mainly to operating on elements) showing that the student does not need to relate either the elements or the operations to physical reality in order to work with them. This ability to work without the requirement of uniqueness of the elements allows satisfactory responses to be made to items such as the following where the problem is to decide on the equality or otherwise of pairs of expressions such as:

(1) $a + b$ and $(a + 1) + (b + L)$

(2) $a + b$ and $(a + 1) + (b - 1)$

(3) $(a + 1) \times (b - 1)$ and $(a - 2) \times (b + 1)$

(4) $(a - b) \times (a + b)$ and $(a - a) \times (b \times b)$

The level of difficulty here is a function not only of the degree of abstraction of the elements but also of the structure of the operations themselves.

Typically responses at this level show that the students can accept lack of closure and are capable of dealing with variables as such because they can hold back from drawing a final conclusion until they have considered various possibilities, an essential strategy for determining a relationship as distinct from obtaining a unique result. For instance,

given V = L x W x H they would not only be able to obtain unique results by appropriate substitutions in the formula but would also be able to discuss meaningfully the effect of various transformations on the formula, e.g., what would you predict for "V" if "L" was increased, "W" decreased and "H" held constant?

The inverse process is seen as an operation which is used to balance or compensate without necessarily affecting the existence of an earlier operation. As such it possesses a reciprocal quality. For example, an item such as

if (p * r) * q = (a * b) * q

then p * r = a * b

can be handled (the elements p, r, q, a, b and the operation "*," with its inverse, being suitably defined).

Responses at this level show that the student is able to see meaning in, and work within, an arbitrary well-defined but unfamiliar system. This ability allows the students to look on the propositions and conditions themselves as the reality, and not to require a link with the physical world prior to working on a problem where the system is defined. They are able to work within a simple mathematical system and are capable of seeing inconsistencies in such a system.

In summary then, extended abstract responses have the following characteristics: acceptance of lack of closure, the use of the reciprocal operation, and the ability to work with multiple interacting and abstract systems. All these characteristics involve a comprehensive use of the given data together with related hypothetical constructs.

Examples of Students' Responses in Two Particular Mathematical Items

In the previous subsection the general aspects of the SOLO characteristics for different levels of functioning with mathematical material have been described. In a number of topics and/or concepts which are incorporated in most mathematics curricula, either implicitly or explicitly, sufficient research has been done to fill out the general summary given above. I have produced three major reports (Collis, 1975a, b; Collis & Biggs, 1979) to which the interested reader might refer. In this section I will give examples of responses at each level for two different elementary mathematical items. Responses were taken from protocols obtained from children of various ages, interviewed individually. A summary of the first protocol has been referred to in explaining some concepts in earlier sections of this chapter.

Example 1.

Question: "What is the value of $\square$ in $\square + 4 = 9$?"

Prestructural

S: "10."

E: "Can you tell me how you got 10?"

S: "10 is my favorite number!"

Comment: irrelevant response; refusal to get involved in data given.

Unistructural

S: "5."

E: "Can you tell me how you got 5?"

S: "5 and 4 are 9."

E: "Could you do it another way?"

S: "No."

E: "Would it help if I did this,

$\square = 9 - 4$?"

S: "No - that's a different sum."

Comment: direct forward closure, using table facts or counting and refusal to accept that subtraction could be related in any way to an addition statement.

Multistructural

S: "5."

E: "Can you tell me how you get 5?"

S: "5 and 4 are 9."

E: "Could you do it another way?"

S: "Yes, take 4 from 9."

E: "You mean you can do it like this, $\square = 9 - 4$?"

S: "Yes."

E: "Why does that work?"

S: "Because it gives you the answer."

Comment: sees that subtraction is a useful procedure for getting the answer but it is simply another way and has no relation to a reversible universe notion.

Relational

S: "5."

E: "Can you tell me how you get 5?"

S: "9 subtract 4 gives 5."

E: "You mean you can do it like this, $\square = 9 - 4$?"

S: "Yes."

E: "Why does that work?"

S: "Well, you've got this number, □, which when you add it to 4 gives you 9. Thus if you want to find the other number you take 4 from 9.

E: "You are telling me then that I can set out my working thus:

□ + 4 = 9 - ①

□ = 9 - 4 - ②."

S: "Yes."

E: "Now ① & ② don't look very much alike to me. Can you help me with any other explanation?"

After some discussion which hinges on S giving his original explanation in different ways, E makes a suggestion.

E: "Putting in another line might help me - would this help you

□ + 4 = 9

□ + 4 - 4 = 9 - 4?"

S: (After a considerable pause and careful consideration.) "No, you're just going to do a lot of work for nothing; it ain't going to get you nowhere!"

Comment: sees the inverse relationship existing between addition and subtraction and justifies the response quite correctly within the constraints of the closed system. Does not see any point in the suggestion for balancing both sides of the equation.

Extended Abstract

S: "5."

E: "Can you tell me how you get 5?"

S: "9 subtract 4 gives 5."

E: "You mean you can do it like this, □ = 9 - 4?"

S: "Yes."

E: "I do not see much connection between □ + 4 = 9 and □ = 9 - 4; they don't look at all alike. Can you give me an explanation?"

S: (Without hesitation and without previous instruction S proceeds as below explaining as he goes.)

$$\square + 4 = 9$$
$$\square + 4 - 4 = 9 - 4$$
$$\square + 0 = 5 + 4 - 4$$
$$\square = 5 + 0$$
$$\square = 5$$

Comment: When pressed, sees the original statement as a balanced logical system and proceeds to operate by appealing to appropriate logico-mathematical principles.

In this first example I have given almost verbatim the interchanges between the experimenter and the student. The same kinds of interchanges were involved in the next example but I have omitted the full details and only set forth the significant student responses at each level. It needs to be borne in mind at this point that the level of response is not necessarily age/grade related. Although the individual's stage of cognitive development probably sets the upper limit to the level of response, it certainly does not set the lower bound.

Example 2.

Question: "What is the value of $\square$ in the following statement: $(72 \div 36) \times 9 = (72 \times 9) \div (\square \times 9)$?"

Prestructural

Ss: (i) "Can't do it";
(ii) "Haven't learnt those yet";
(iii) "I don't like long sums."

Comment: Ss give an irrelevant response or refuse to get engaged in the problem.

Unistructural

Ss: (i) "36 - there's no 36 on the right hand side."
(ii) "$72 \div 36 = 2$" - S gives up at this point.

Comment: Ss responses usually indicate low level pattern seeking or one closure and then the task is given up.

Multistructural

Ss: $(72 \div 36) \times 9 = (72 \times 9) \div (\square \times 9)$
$2 \times 9 \quad 648 \div (\square \times 9)$
$18 \quad 648 \div ? = 2$ i.e. 324
18
Hence $\square = 324$

Comment: Ss perform a series of closures but lose the thread of what they are after and get mixed up. In the above example S closed off the left hand side to "18" without any trouble. The problem arose on the right hand side - "18" was required, "9" was available, where was a "2" to come from? - hence the mix-up.

Relational

Ss: $(72 \div 36) \times 9 = (72 \times 9) \div (\square \times 9)$
$2 \times 9 \quad 648 \div (\square \times 9)$
$18 \quad 648 \div 9 = 72$
then $72 \div 4 = 18$
Hence $\square = 4$

Comment: Ss perform a series of closures but are able to keep the overall picture before them so that they do not get mixed-up at a crucial stage in the process.

Extended Abstract

Ss: $(72 \div 36) \times 9 = (72 \times 9) \div (\square \times 9)$
$(72 \div 36) \times 9 = (72 \times 9) \div (\square \times 9)$
$\left[\frac{a}{b} \times y = \frac{ay}{b}\right]$

thus:
$(72 \div 36) \times 9$
$= (72 \times 9) \div 36$
$= (72 \times 9) \div (4 \times 9)$
Hence $\square = 4$

Comment: The initial thinking is in terms of examining the relationships. These suggest looking at some reassociation of the elements concerned. No closures are undertaken; all the work is done by focusing on the abstract logic of the operations and a complete overview of the problem.

Conclusion

It should be clear that focusing the educator's attention on response levels and their structure has major implications for the educational process which takes place in the classroom (Collis & Biggs, 1979). First, it takes the emphasis off the level of cognitive development and/or IQ, about which the individual teacher can do very little. Instead, attention is drawn to the structure of the child's response which, if the teacher is able to recognize it, enables a dialogue to begin with a view to raising the child's level of responding - surely a desirable classroom aim regardless of the content matter involved.

Apart from its use in formative evaluation as indicated, the response model gives the teacher a rational basis for normal evaluation of the child's level of functioning in an area. It indicates clearly that in mathematics the correct answer can be obtained in a number of ways and is not in itself sufficient to tell the teacher how well the child understands what he or she is supposed to be doing.

Much work needs to be done in mathematics education before the full benefits of focusing attention on the structure of a child's responses can be attained. One field where the concept can be put to immediate use with good effect would be curriculum development. A series of normative studies would indicate the expectations which the teacher might have of a normal age/grade group in terms of response complexity, together with specifying the structure of the next level of complexity. This

knowledge would be of enormous benefit to schools and teachers involved in school-based curriculum development.

Finally, the notions explicated above would suggest that we investigate closely the initial understanding of addition and subtraction which the child begins school. Children do a great deal of learning before they come to school and indeed develop quite sound problem solving skills appropriate to their own level of cognitive functioning. It seems quite illogical to ignore this and begin by imposing a potentially more efficient adult model on the children - a model which the children have not the cognitive capacity to come to terms with, as some of the research referred to in this paper shows.

Children might well see the structure of the following problem as a "take-away": "Mary has 8 balloons and she gives 5 to Mark. How many has she left?" On the other hand, children might view the following problem as some kind of "matching": "Mary has 8 balloons, Mark has 5. How many more balloons has Mary than Mark?" For the teacher to categorize these as both mere applications of the subtraction algorithm could well be one of the first steps towards teaching children that mathematics is not something <u>one does</u> or <u>thinks with</u> but a sequence of unconnected school-valued skills which, they are led to believe, will be of use later.

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